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Photon rockets and the Robinson-Trautman geometries

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Abstract

We point out the relation between the photon rocket spacetimes and the Robinson Trautman geometries. This allows a discussion of the issues related to the distinction between the gravitational and matter energy radiation that appear in these metrics in a more geometrical way, taking full advantage of their asymptotic properties at null infinity to separate the Weyl and Ricci radiations, and to clearly establish their gravitational energy content. We also give the exact solution for the generalized photon rockets.

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1 Introduction

Examples of spacetimes representing the so called photon rockets, have been recently studied in the literature[1][2]. An important problem under consideration in those examples was the existence of gravitational radiation associated with the accelerated motion of the 'rocket', and its relation with the outgoing energy momentum, in the form of a 'null (or photon) fluid', that accompanies the accelerated rocket motion. In the example of reference [1], which is based on the Kinnersley[3] metric, the well-known result of the absence of gravitational radiation is analyzed as an application of the quadrupole formula. A problem in this type of analysis is the presence of singularities in the metric, precisely along the 'world-line' of the 'accelerated' object. To analyze this problem, and

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be able to assign a meaning to concepts such as 'rocket trajectory' and 'distributional energy momentum tensor', in reference [2] it is considered the associated linearized problem, and its relation to the exact solution of Einstein equation; obtaining conditions for the presence of a non-vanishing flux of gravitational radiation.

In this note we present a different approach. We start by pointing out the relation between the photon rocket spacetimes and the Robinson-Trautman geometries. All these metrics are asymptotically flat. This allows us to present a more geometrical discussion of the issues related with the distinction between the gravitational and matter-energy radiation that appear in these metrics, taking full advantage of their asymptotic properties at future null infinity to separate the Weyl and Ricci radiations, and to clearly establish their gravitational energy content in the general case. We show in the last section that the exact solutions to the generalized photon rockets are the Robinson-Trautman geometries.

2 The Robinson-Trautman geometries

The vacuum solutions containing a congruence of diverging null geodesics, with vanishing shear and twist, were first studied by Robinson and Trautman[4]. These metrics can be expressed[5] in terms of the following line element

$$ds^2 = \left(-2Hr + K - 2\frac{m(u)}{r}\right) du^2 + 2\, du\, dr - \frac{r^2}{P^2} d\zeta\, d\bar{\zeta} \qquad (1)$$

where $P = P(u, \zeta, \bar{\zeta})$, $H = \frac{\dot{P}}{P}$, $K = \Delta \ln P$, a doted quantity denotes its time derivative and Δ is the two-dimensional Laplacian for the two-surfaces $u = constant$, $r = constant$ with line element

$$dS^2 = \frac{1}{P^2}\, d\zeta\, d\bar{\zeta}.$$

It is usually convenient to describe this line element in terms of the line element of the unit sphere; this is done by expressing P in terms of $P = V(u, \zeta, \bar{\zeta}) P_0(\zeta, \bar{\zeta})$, where P_0 is the value of P for the unit sphere. If l denotes the vector field that generates the null congruence, then $l = du$, $l(r) = 1$, $l(\zeta) = 0$ and $l(\bar{\zeta}) = 0$. In other words this is the coordinate system adapted to the geometry.

In reference [4] it was found that the vacuum Einstein equation can be reduced to a parabolic equation for a scalar depending on three variables, the so called Robinson-Trautman equation; which using the GHP[6] notation has the form

$$-3\, m\, \dot{V} = V^4\, \eth^2 \bar{\eth}^2 V - V^3 \eth^2 V \bar{\eth}^2 V; \qquad (2)$$

where we fixed the freedom of redefining u in such a way that $m(u)$ is actually a constant, and $\eth$ is the GHP edth operator of the unit sphere. We will refer to a

line element with V satisfying this equation as a Robinson-Trautman solution. On the other hand, if the Robinson-Trautman equation is not required, then the solution is no longer vacuum and there is only one component of the Ricci tensor different from zero, given by

$$\Phi_{22}^{(RT)} = \frac{-3\, m\frac{\dot{V}}{V} - V^3 \eth^2 \bar{\eth}^2 V + \eth^2 V \bar{\eth}^2 V}{r^2}; \tag{3}$$

where the (RT) is to emphasize the fact that in this case we are using the null tetrad adapted to the null congruence. We refer to this as a Robinson-Trautman geometry.

Let us consider next some examples of theses geometries.

3 First example: flat spacetime

As an example, let us consider the case in which one requires a Robinson-Trautman geometry to be flat; which demands, among other things, m to be zero. In this case the above equation describes the Minkowski line element in terms of a set of null polar coordinates associated with a timelike curve. The scalar $V(u,\zeta,\bar{\zeta})$, in this case, contains only $Y_{00}(\zeta,\bar{\zeta})$ and $Y_{1m}(\zeta,\bar{\zeta})$ spherical harmonics and we will denote it by $V_I(u,\zeta,\bar{\zeta})$. The line element becomes

$$ds_0^2 = \left(-2\frac{\dot{V}_I}{V_I} r + 1\right) du^2 + 2\, du\, dr - \frac{r^2}{V_I^2 P_0^2} d\zeta\, d\bar{\zeta}. \tag{4}$$

It is a well-known result (see for example ref. [7]) that under these conditions the vector l is singular on a timelike world-line $\gamma(\tau)$, which can be determined by the coordinate condition $r = 0$.

4 Second example: massive 'particle' with arbitrary motion

Let us now take the previous line element, associated to the timelike curve $\gamma(\tau)$, and add a term of the form $\frac{-2m}{r} l_a l_b$; that is

$$ds^2 = ds_0^2 - \frac{2m}{r}\, du^2; \tag{5}$$

which can also be thought of as a Robinson-Trautman geometry where V is restricted to the condition $\eth^2 V = 0$. We use Latin indices to denote abstract indices. Under this condition we distinguish between two cases:

Geodesic motion: When the curve $\gamma(\tau)$ is a geodesic, the resulting line element is the Schwarzschild metric, written in a different coordinate system.

Non-geodesic motion: If the curve $\gamma(\tau)$ is not a geodesic, the Ricci tensor is different from zero. From the above equation one can deduce that

$$R_{ab} = \frac{6m\dot{V}}{Vr^2} l_a l_b. \tag{6}$$

We recognize this as the case of the 'photon rocket'[3].

5 Radiation in the general case

The general Robinson-Trautman geometry is asymptotically flat. This can be seen from the fact that the coordinate r is proportional to the luminosity distance, and therefore the conformal factor

Ω , used in the discussion of the asymptotic behavior, can be taken just as r^{-1} ; then it is deduced from references [8] and [9] that the Weyl tensor behaves as Ω, in the vicinity of future null infinity, and from equation (3) that the Ricci tensor behaves respectively as Ω^2 ; which implies that the Riemann tensor goes to zero as Ω as one approaches future null infinity, assuring the asymptotic flatness behavior[8] of the geometry. This picture could be generalized to consider the case of singular behavior for the scalar V on future null infinity; to admit the case, for example, of a photon rocket reaching this region; this would force us to consider the notion of asymptotic flatness with future null infinities which are not complete. This generalization, however, does not introduce any relevant issue for the following discussion.

It has been proved[8] that the most general class of asymptotically flat spacetimes have the BMS structure at future null infinity. The BMS symmetry group has an infinite dimensional Lie algebra with a four-dimensional normal subgroup[10], the so called translation subgroup. This fact permits one to construct unambiguously the Bondi momentum in regular[1] spacetimes. It has also been proved that any asymptotically flat spacetime[8] shows the radiation behavior found in regular spacetimes; that is the radiation content of an asymptotically flat spacetime is also an unambiguous concept.

One can further prove that in a regular asymptotically flat spacetime, the radiation content is associated with the time variation of the Bondi momentum. More concretely, the Bondi momentum can be given in terms of the GHP formalism by

$$P^\alpha = -\frac{1}{4\pi}\int \left(\Psi_2^0 + \sigma^0 \frac{\partial \bar{\sigma}^0}{\partial u_B}\right) \hat{l}^\alpha \, dS^2 = \frac{1}{4\pi}\int \frac{m}{V^3} \hat{l}^\alpha \, dS^2 \tag{7}$$

where we are using a Bondi frame adapted to the Bondi coordinates $(u_B, \zeta, \bar{\zeta})$, $\alpha = 0, 1, 2, 3$ and

$$\left(\hat{l}^\alpha\right) = \left(\sqrt{4\pi}\, Y_{00},\ -\sqrt{\frac{2\pi}{3}}\ (Y_{11} - Y_{1-1})\, ,\ i\sqrt{\frac{2\pi}{3}}\,(Y_{11} + Y_{1-1})\, ,\ \sqrt{\frac{4\pi}{3}}\ Y_{10}\right).$$

[1]By a regular asymptotically flat spacetime we mean an at least twice differentiable conformal structure at future null infinity.

The Weyl radiation behavior is encoded in the asymptotic components Ψ_4^0 and Ψ_3^0 , which in turn can be expressed in terms in a the Bondi frame by its shear, i.e.: $\Psi_4^0 = -\partial^2\bar{\sigma}^0/\partial u_B^2$ and $\Psi_3^0 = -\eth(\partial\bar{\sigma}^0/\partial u_B)$. The Ricci radiation behavior is in turn encoded in the asymptotic component Φ_{22}^0.

The time variation of the Bondi momentum in terms of the Bondi time is given by

$$\frac{dP^\alpha}{du_B} = -\frac{1}{4\pi}\int\left(\frac{\partial\sigma^0}{\partial u_B}\frac{\partial\bar{\sigma}^0}{\partial u_B} - \Phi_{22}^0\right)\hat{l}^\alpha\ dS^2; \tag{8}$$

while the variation of the Bondi momentum in terms of the Robinson-Trautman time is

$$\frac{dP^\alpha}{du} = -\frac{1}{4\pi}\int\left(\frac{\eth^2 V\bar{\eth}^2 V}{V} - \frac{\Phi_{22}^{(RT)0}}{V^3}\right)\hat{l}^\alpha\ dS^2 \tag{9}$$

In the present case of the Robinson-Trautman geometry the Bondi shear is given by

$$\frac{\partial\sigma^0}{\partial u_B} = \frac{\eth^2 V}{V}$$

and the Ricci Φ_{22}^0 component is given by

$$\Phi_{22}^0 = \frac{\Phi_{22}^{RT}}{V^4}.$$

Note that there is no contradiction among the last two equations and (8) and (9), since $\partial u_B/\partial u = V$.

6 The photon rockets

A family of axisymmetric metrics corresponding to a particle emitting null fluid anisotropically, and therefore accelerating, the so called photon rockets, first described by Kinnersley[3], has been recently reanalyzed in reference [1]. These metrics actually coincide with the line element appearing in equation (5). Using this Kerr-Schild decomposition, the corresponding energy momentum tensor was studied in reference [1] to estimate the matter flux(there called radiation), over a local two-sphere. It can be seen from the above presentation that the unambiguous radiation is related to the change of momentum by equation (8) which under this circumstances adopts the form

$$\frac{dP^\alpha}{du_B} = \frac{1}{4\pi}\int\Phi_{22}^0\hat{l}^\alpha\ dS^2 = -\frac{1}{4\pi}\int\frac{3\ m\ \dot{V}}{V^5}\hat{l}^\alpha\ dS$$

since in this case $\partial\sigma^0/\partial u_B = 0$ due to the fact that $\eth^2 V = 0$. In other words, this spacetime does not show gravitational radiation, i.e. Weyl radiation, and there is only matter radiation.

In reference [2] the photon rocket spacetimes where studied from the perspective of linearized gravity. The problem could be stated in the following terms; find the linearized solution to Einstein equation with the source

$$T_{ab} = \frac{w(u,\zeta,\bar{\zeta})}{4\pi\, r^2}\, l_a\, l_b \tag{10}$$

where[2] l_a is the generator of a congruence of null geodesics emanating from $r = 0$. More generally, the energy momentum tensor of equation (10) can naturally be considered the source for the generalized photon rockets. Equation (3.12) of reference [2] establishes that the source must satisfy the following equation of motion:

$$\frac{d\,(M(u)\, u^a)}{du} \equiv \frac{d\,(P^a)}{du} = -\frac{1}{4\pi}\int w\, l^a d\Omega$$

which can be seen to agree with our equation (9) above by recognizing that the vector k^a of reference [2] is identified with our vector $-l^a$; and that $w = -\Phi_{22}^{(RT)}$; $l^a = \hat{l}^a/V$ and $d\Omega = dS^2/V^2$; since $\eth^2 V = 0$ for the photon rockets. Note that a relation between $M(u)$ and m is easily obtained by noting that $M(u)^2 = P^\alpha\, P_\alpha$ with P^α given by eq. (7) above. It is worth while to note that although we have chosen m to be a constant, by reparametrizing the coordinate u, we could have taken any other choice, as for example the total (Bondi) mass at the retarded time u, as used in reference [2]. Our choice simplifies the reading of eq. (2), which would otherwise include another term involving the time derivative of m.

It is important to emphasize that while the last equation was deduced from the local properties of the source, eq. (9) was obtained from the asymptotic structure. It happens that in this particular case there is a clear map from inner data to asymptotic data.

The information contained in eq. (3.12) of reference [2] coincides with that obtained from eqs. (30) and (31) of reference [1].

The linearized solutions for the case of a general source $w(u,\zeta,\bar{\zeta})$ involve instead very complicated expressions, as for example equation (4.35) of reference [2]. We would like to point out here that the problem of the generalized photon rockets (i.e. with the source (10)) has exact solutions which are the Robinson-Trautman geometries. This can be seen by comparing the Ricci tensor expressed by the component shown in eq. (3) with the energy tensor of eq. (10), from which it is deduced that an exact solution is recognized if one identifies w with

$$w(u,\zeta,\bar{\zeta}) = 3m\frac{\dot{V}}{V} + V^3\eth^2\bar{\eth}^2 V - V^2\eth^2 V\bar{\eth}^2 V. \tag{11}$$

In particular, the photon rocket geometries are those for which w is constructed with V satisfying $\eth^2 V = 0$. Note that equation (11) can be read in two ways: one in which $V(u,\zeta,\bar{\zeta})$ is given and one computes $w(u,\zeta,\bar{\zeta})$, thus providing an exact solution for the source (10); and the other in which the scalar

[2] They use the Greek letter epsilon instead of w in that reference.

w is given and one has to look for the corresponding V. The latter is a problem whose solution is outside the scope of the present article.

The Robinson-Trautman geometries were also studied in reference [9] from the point of view of their Petrov classification.

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